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Three-manifold invariants and their relation with the fundamental group

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Abstract. We consider the 3-manifold invariant $I(M)$ which is defined by means of the Chern-Simons quantum field theory and which coincides with the Reshetikhin-Turaev invariant. We present some arguments and numerical results supporting the conjecture that, for nonvanishing $I(M)$, the absolute value $|\, I(M) \,|$ only depends on the fundamental group $\pi_1(M)$ of the manifold M. For lens spaces, the conjecture is proved when the gauge group is $SU(2)$. In the case in which the gauge group is $SU(3)$, we present numerical computations confirming the conjecture.

1 Introduction

Recently, new 3-manifold invariants [1, 2] have been discovered; the algebraic aspects of their construction, which is based on the structure of simple Lie groups, are well understood [3, 4, 5, 6, 7, 8, 9]. However, the topological meaning of these invariants is still unclear. Let us denote by $I(M)$ the invariant of the 3-manifold M which is closed, connected and orientable; $I(M)$ is the invariant defined by means of the Chern-Simons quantum field theory [1, 9] and coincides with the Reshetikhin-Turaev invariant [2, 3]. In general, it is not known how $I(M)$ is related, for instance, to the homotopy class of M or to the fundamental group of M. In this article we shall formulate the following

> Conjecture: for nonvanishing $I(M)$, the absolute value $|\,I(M)\,|$ only depends on the fundamental group $\pi_1(M)$.

In the absence of a general proof, we shall verify the validity of the conjecture for a particular class of manifolds: the lens spaces. There are examples of lens spaces M_1 and M_2 with the same fundamental group $\pi_1(M_1) \simeq \pi_1(M_2)$ which are not homeomorphic; for all these manifolds, we shall prove that (for nonvanishing invariants) $|\,I(M_1)\,| = |\,I(M_2)\,|$ when $I(M)$ is the invariant associated with the group $SU(2)$. In the case in which the gauge group is $SU(3)$, we will present numerical computations confirming the conjecture. Our results are in agreement with the computer calculations for $SU(2)$ of Freed and Gompf [10] and the expression of the $SU(2)$ invariant obtained by Jeffrey [11]. Differently from [10] and [11], our approach is based exclusively on the properties of 3-dimensional Chern-Simons quantum field theory. We shall use general surgery rules to compute $I(M)$ and, in our construction, invariance under Kirby moves is manifestly satisfied.

Our notations and conventions are described in section 2. The expression of the invariant $I(M)$ for a generic lens space is derived in section 3 and, for the gauge group $SU(2)$, the validity of our conjecture is proved in section 4. The numerical computations for the group $SU(3)$ are reported in section 5 and the conclusions are contained in section 6.

2 Surgery rules

The basic ingredient in the construction of the 3-manifold invariant $I(M)$ is a polynomial invariant $E(\mathcal{L})$ for oriented, framed and coloured links $\{\,\mathcal{L}\,\} \subset S^3$. In the Chern-Simons field theory, this link invariant is defined by the expectation values of the Wilson line operators [1]; each link component is framed and its colour is given by an irreducible representation of a simple compact Lie group which is called the gauge group. For example, when the gauge group is $SU(N)$ and each link component has colour corresponding to the fundamental representation of $SU(N)$, $E(\mathcal{L})$ is determined by the skein relation [1, 12]

$$q^{1/(2N)}\, E(\mathcal{L}_+) \,-\, q^{-1/(2N)}\, E(\mathcal{L}_-) \;=\; (\, q^{1/2} - q^{-1/2}\,)\, E(\mathcal{L}_0) \qquad , \tag{1}$$

where $q = \exp(-i2\pi/k)$ is the deformation parameter and k is the renormalized coupling constant of the Chern-Simons field theory. The standard skein-related links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ correspond to a configuration with over-crossing, under-crossing and no-crossing respectively. Moreover, under an elementary ± 1 modification of the framing of a link component, $E(\mathcal{L})$ gets multiplied by the factor $q^{\pm(N^2-1)/2N}$. Finally, the factorization property [1, 12] which holds for the distant union of links fixes the normalization of the unknot with preferred framing

$$E_0[\,\text{fund.}\,] \;=\; (\,q^{N/2} - q^{-N/2}\,)/(\,q^{1/2} - q^{-1/2}\,) \qquad . \tag{2}$$

In general, the colour which characterizes one link component is an element of the algebra $\mathcal{T}$ which coincides with the complex extension of the representation ring of the gauge group. The sum operation in this algebra extends by linearity to $E(\mathcal{L})$; whereas the product operation in the colour algebra $\mathcal{T}$ simply corresponds to the satellites obtained from the companion links by standard cabling [6, 13]. For unitary groups, the fundamental skein relation (1), the normalization (2) of the unknot and the correspondence between cabled components and higher-dimensional representations of the gauge group uniquely determine the values of the link invariant $E(\mathcal{L})$ for arbitrary coloured link components.

Let us denote by $\mathcal{L}_1 \# \mathcal{L}_2[\rho]$ the connected sum of the links $\mathcal{L}_1$ and $\mathcal{L}_2$ in which the component which connects these two links has colour given by the irreducible representation ρ of the gauge group. From the properties of the Chern-Simons field theory it follows that [1, 13]

$$E(\,\mathcal{L}_1 \# \mathcal{L}_2[\rho]\,) \;=\; \frac{E(\mathcal{L}_1)\,E(\mathcal{L}_2)}{E_0[\,\rho\,]} \qquad , \tag{3}$$

where $E_0[\,\rho\,]$ is the value of the unknot with preferred framing and colour ρ.

For integer values of the Chern-Simons coupling constant k ($k = 1, 2, 3, ...$), the set of vanishing link invariants defines an ideal $\mathcal{I}_k$ of $\mathcal{T}$. Thus, for fixed integer k, the colour states belong to the algebra [13] of the equivalence classes

$$\mathcal{T}_{(k)} \;=\; \mathcal{T} \,/\, \mathcal{I}_k \qquad . \tag{4}$$

Usually, $\mathcal{T}_{(k)}$ is of finite order [14] and, for appropriate values of k, $\mathcal{T}_{(k)}$ is isomorphic with the Verlinde algebra [15] which is determined by of the fusion rules of certain two-dimensional conformal models [1]. We shall now concentrate on $\mathcal{T}_{(k)}$ when the gauge group G is $SU(2)$ [13] or $SU(3)$ [16]. For $G = SU(2)$ and $k = 1$, $\mathcal{T}_{(1)}$ is isomorphic with the group algebra of Z_2, which is the center of $SU(2)$. For $G = SU(2)$ and $k \geq 2$, the ideal $\mathcal{I}_k$ is generated by the representation with $J = (k-1)/2$ and $\mathcal{T}_{(k)}$ is of order $(k-1)$. For $G = SU(3)$ and $k = 1, 2$, the algebra (4) is isomorphic with the group algebra of Z_3, which is the center of $SU(3)$. For $G = SU(3)$ and $k \geq 3$, the ideal $\mathcal{I}_k$ is generated by the two irreducible representations with Dynkin labels $(k-1, 0)$ and $(k-2, 0)$; in this case, $\mathcal{T}_{(k)}$ is of order $(k-1)(k-2)/2$.

We shall denote by $\{\,\psi[\,i\,]\,\}$ (with $i = 1, 2, ..., \dim(\mathcal{T}_{(k)})$) the elements of a basis in $\mathcal{T}_{(k)}$. When $G = SU(2)$ and $k \geq 2$ or when $G = SU(3)$ and $k \geq 3$, each

$\psi[\,i\,]$ represents the equivalence class of an irreducible representation of the gauge group. For low values of k , $\psi[\,i\,]$ corresponds to an irreducible representation of the gauge group up to a nontrivial multiplicative factor [13, 16]. The unit in $\mathcal{T}_{(k)}$ will be denoted by $\psi[1]$; $\psi[1]$ is the class defined by the trivial representation.

Let us now consider the definition of the 3-manifold invariant $I(M)$. Each 3-manifold M , which is closed, connected and orientable, admits a surgery presentation [17] given by Dehn surgery on S^3 . Each "honest" [17] surgery instruction can be represented by a framed link $\mathcal{L} \subset S^3$ with components $\{\,\mathcal{L}_b\,\}$ with $b = 1, 2, \ldots$. The surgery link $\mathcal{L}$ is not oriented and an integer surgery coefficient r_b is attached to the component $\mathcal{L}_b$. The framing $\mathcal{L}_{bf}$ of $\mathcal{L}_b$ is specified by the linking number

$$\ell\mathrm{k}\,(\mathcal{L}_b, \mathcal{L}_{bf}) \;=\; r_b \quad . \tag{5}$$

The surgery link associated to the manifold M is not unique. Indeed, if the surgery links $\mathcal{L}$ and $\mathcal{L}'$ are related by a finite sequence of Kirby moves, the corresponding manifolds are homeomorphic [18]. Therefore, each 3-manifold M is characterized by a class of "equivalent" surgery links in S^3 , where "equivalent" links means links related by Kirby moves.

Let $\mathcal{L} \subset S^3$ be a surgery link for the manifold M . The invariant $I(M)$ is defined in terms of the expectation value $E(\mathcal{L})$ of the Wilson line operators associated with the surgery link $\mathcal{L}$. More precisely, one introduces an (arbitrary) orientation and a particular colour state Ψ_0 for each component of $\mathcal{L}$. For fixed integer k , the surgery colour state $\Psi_0 \in \mathcal{T}_{(k)}$ is [2]

$$\Psi_0 \;=\; a_k \sum_i E_0[\,i\,]\,\psi[\,i\,] \qquad , \tag{6}$$

where the sum is performed over all the elements $\{\,\psi[\,i\,]\,\}$ of the basis of $\mathcal{T}_{(k)}$. The coefficients $\{\,E_0[\,i\,]\,\}$ coincide with the expectation values of the unknot with preferred framing and colour $\psi[\,i\,]$. When the gauge group G is $SU(2)$, a_k is given by [13]

$$a_k \;=\; \begin{cases} 1/\sqrt{2} & k \;=\; 1 \\ \sqrt{\frac{2}{k}}\,\sin\,(\pi/k) & k \;\geq 2 \quad ; \end{cases} \tag{7}$$

whereas, when $G = SU(3)$, one has [16]

$$a_k \;=\; \begin{cases} 1/\sqrt{3} & k \;=\; 1,2 \\ 16\cos(\pi/k)\sin^3(\pi/k)/(k\sqrt{3}) & k \;\geq 3 \quad . \end{cases} \tag{8}$$

We shall denote by $\sigma(\mathcal{L})$ the signature of the linking matrix associated with $\mathcal{L}$; $\sigma(\mathcal{L})$ does not depend on the choice of the orientation of $\mathcal{L}$. Let us define the function $I(\mathcal{L})$ by means of the relation [2]

$$I(\mathcal{L}) \;=\; \exp\,[\,i\,\theta_k\,\sigma(\mathcal{L})\,]\;\; E(\mathcal{L}) \qquad , \tag{9}$$

where, for $G = SU(2)$, the phase factor $e^{i\theta_k}$ is [2, 13]

$$e^{i\theta_k} \;=\; \begin{cases} \exp\,(-i\pi/4) & k = 1 \\ \exp\,[i\pi 3(k-2)/(4k)] & k \geq 2 \quad ; \end{cases} \tag{10}$$

and, for $G = SU(3)$, the phase factor is [16]

$$e^{i\theta_k} = \begin{cases} \exp(i\pi/2) & k = 1 \\ \exp(-i\pi/2) & k = 2 \\ \exp(-i6\pi/k) & k \geq 3 \end{cases} \quad . \tag{11}$$

It can be verified [2, 3, 13, 16] that $I(\mathcal{L})$ is invariant under Kirby moves and then it represents a topological invariant for the 3-manifold M. In what follows, we shall denote this invariant by $I(M)$.

It should be noted that the multiplicative phase factor in (9) is not a matter of convention (or choice of framing); the presence of the term $\exp[\, i\,\theta_k\, \sigma(\mathcal{L})\,]$ in (9) guarantees the invariance of $I(\mathcal{L})$ under Kirby moves. According to the definition (9), the normalization of the 3-manifold invariant $I(M)$ is fixed by $I(S^3) = 1$.

In order to compare $I(M)$ with the expressions obtained in [10, 11], we need to produce the relation between the link invariants and the representation matrices of the mapping class group of the torus.

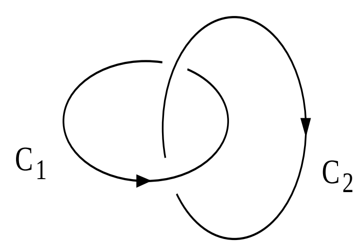

Figure 1

Let us consider the Hopf link in S^3, shown in Figure 1; let the two link components C_1 and C_2 have preferred framings and colours $\psi[\,i\,]$ and $\psi[\,j\,]$ respectively. The associated Chern-Simons expectation value is denoted by

$$H_{ij} = E(\, C_1, \,, \psi[\,i\,]\,; C_2, \,, \psi[\,j\,]\,) \quad . \tag{12}$$

The complex numbers $\{\, H_{ij}\,\}$ where $i, j = 1, 2, ..., \dim(\mathcal{T}_{(k)})$ can be understood as the matrix elements of the so-called Hopf matrix H. Note that H is symmetric and that $E_0[\,i\,] = H_{1i} = H_{i1}$. Let $Q(i)$ be the value of the quadratic Casimir operator of the irreducible representation of the gauge group which is associated with an element of the class $\psi[\,i\,]$. One can show [14] that the matrices

$$X_{ij} = a_k\, H_{ij} \quad ; \quad Y_{ij} = q^{Q(i)}\, \delta_{ij} \quad ; \quad C_{ij} = \delta_{ij^*} \quad . \tag{13}$$

give a projective representation of the modular group

$$X^2 = C \quad ; \tag{14}$$

$$(X\, Y)^3 \;=\; e^{-\,i\,\theta_k}\, C \qquad . \tag{15}$$

This representation is isomorphic with the representation obtained in two-dimensional conformal field theories [1]; X corresponds to the S matrix of the conformal models and Y is the analogue of the T matrix.

3 Lens Spaces

Lens spaces, which are characterized by two integers p and r, will be denoted by $\{ L_{p/r} \}$. The fundamental group of $L_{p/r}$ is Z_p. Two lens spaces $L_{p/r}$ and $L_{p'/r'}$ are homeomorphic if and only if $|p| = |p'|$ and $r = \pm r' \pmod p$ or $rr' = \pm 1 \pmod p$. Thus, we only need [17] to consider the case in which $p > 1$ and $0 < r < p$; moreover, r and p are relatively prime. The lens spaces $L_{p/r}$ and $L_{p'/r'}$ are homotopic if and only if $|p| = |p'|$ and $rr' = \pm$ quadratic residue $(\mathrm{mod}\ p)$. Consequently, one can find examples of lens spaces which are homotopic but are not homeomorphic; for instance, $L_{13/2}$ and $L_{13/5}$. One can also find examples of lens spaces which are not homeomorphic and are not homotopic but have the same fundamental group; for instance, $L_{13/2}$ and $L_{13/3}$.

One possible surgery instruction corresponding to the lens space $L_{p/r}$ is given the unknot [17] with rational surgery coefficient (p/r). From this surgery presentation one can derive [17] a "honest" surgery presentation of $L_{p/r}$ by using a continued fraction decomposition of the ratio (p/r)

$$\frac{p}{r} = z_d - \cfrac{1}{z_{d-1} - \cfrac{1}{\ddots - \frac{1}{z_1}}} \qquad , \tag{16}$$

where $\{z_1,\, z_2, \cdots,\, z_d\}$ are integers. The new surgery link $\mathcal{L}$ corresponding to a "honest" surgery presentation of $L_{p/r}$ is a chain with d linked components, as shown in Figure 2, and the integers $\{z_1,\, z_2, \cdots,\, z_d\}$ are precisely the surgery coefficients.

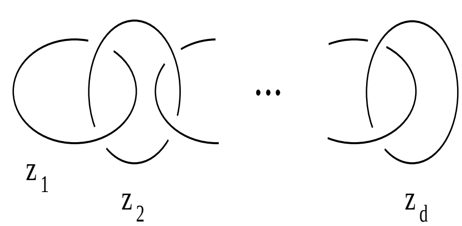

Figure 2

According to the definition (9), the lens space invariant is given by

$$I(L_{p/r}) \;=\; e^{i\theta_k\sigma(\mathcal{L})}\,(\,a_k\,)^d \sum_{j_1,\cdots,j_d\in\mathcal{T}_k}\;\prod_{i=1}^{d}\left(q^{z_i\,Q(j_i)}\right)\;\times$$
$$\times\; E_0[\,j_1]\cdots E_0[\,j_d]\;\; E\,(\,\mathcal{L}\,;\psi[\,j_1]\,,\cdots,\psi[\,j_d])\qquad . \tag{17}$$

The link of Figure 2 can be understood as the connected sum of $(d-1)$ Hopf links $\mathcal{H}$, i.e. $\mathcal{L}=\mathcal{H}\#\mathcal{H}\cdots\#\mathcal{H}$. Therefore, by using equation (3), expression (17) can be written as

$$I(L_{p/r}) \;=\; e^{i\theta_k\sigma(\mathcal{L})}\,(\,a_k\,)^d \sum_{j_1,\cdots,j_d\in\mathcal{T}_k} q^{\left(\sum_{i=1}^d z_i\,Q(j_i)\right)}\; H_{1j_d}\,H_{j_dj_{d-1}}\cdots\, H_{j_2j_1}\,H_{j_11}\quad . \tag{18}$$

In terms of the generators (13) of the modular group, one finds

$$I(L_{p/r}) \;=\; e^{i\theta_k\sigma(\mathcal{L})}\;(a_k)^{-1}\;\left[\,F(p/r)\right]_{11}\qquad , \tag{19}$$

where $[\,F(p/r)]_{11}$ is the element corresponding to the first row and the first column of the following matrix

$$F(p/r) \;=\; XY^{z_d}XY^{z_{d-1}}X\cdots XY^{z_1}X\qquad . \tag{20}$$

The invariant $I(L_{p/r})$ given in equation (19) is in agreement with the expressions obtained in [10, 11] apart from an overall normalization factor.

4 The SU(2) case

In this section, we shall compute $I(L_{p/r})$ for the gauge group $G=SU(2)$. Then, we will show that in this case our conjecture is true; i.e. when $I(L_{p/r})\neq 0$, the absolute value $|\,I(L_{p/r})\,|$ only depends on p.

For $k\geq 2$, the standard basis of $\mathcal{T}_k$ is $\{\,\psi[\,j\,]\,\}$; the index j represents the dimension of the irreducible representation described by $\psi[\,j\,]$ and $1\leq j\leq (k-1)$. The matrix elements of X and Y are

$$(X)_{mn} \;=\; \frac{i}{\sqrt{2k}}\left[\exp\left(-\frac{i\pi mn}{k}\right)\;-\;\exp\left(\frac{i\pi mn}{k}\right)\right]\quad ;$$
$$(Y)_{mn} \;=\; \xi\,\exp\left(-\frac{i\pi m^2}{2k}\right)\delta_{mn}\quad ; \tag{21}$$

with

$$\xi \;=\; \exp(\,i\pi/2k\,)\qquad . \tag{22}$$

When $k=1$, one has

$$X \;=\; \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\quad ,\quad Y \;=\; \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\quad . \tag{23}$$

The algebra $\mathcal{T}_1$ is isomorphic with $\mathcal{T}_3$ and it is easy to verify that

$$I_{k=1}(L_{p/r}) \;=\; \Big[\, I_{k=3}(L_{p/r}) \,\Big]^* \qquad . \tag{24}$$

Therefore, we only need to consider the case $k \geq 2$.

In order to compute $I(L_{p/r})$, we shall derive a recursive relation for the matrix (20); the argument that we shall use has been produced by Jeffrey [11] in a slightly different context. In fact, our final result for $I(L_{p/r})$ is essentially in agreement with the formulae obtained by Jeffrey. Since in our approach the invariance under Kirby moves is satisfied, our derivation of $I(L_{p/r})$ proves that the appropriate expressions given in [10, 11] really correspond to the values of a topological invariant of 3-manifolds.

Let us introduce a few definitions; with the ordered set of integers $\{z_1,\, z_2, \cdots,\, z_d\}$ one can define the following partial continued fraction decompositions

$$\frac{\alpha_t}{\gamma_t} \;=\; z_t \,-\, \cfrac{1}{z_{t-1} \,-\, \cfrac{1}{\ddots - \frac{1}{z_1}}} \qquad , \tag{25}$$

where $1 \leq t \leq d$. The integers α_t and γ_t satisfy the recursive relations

$$\alpha_{m+1} \;=\; z_{m+1}\,\alpha_m \,-\, \gamma_m, \quad , \quad \alpha_1 \;=\; z_1 \quad , \quad \alpha_0 \;=\; 1 \quad ; \tag{26}$$

$$\gamma_{m+1} \;=\; \alpha_m \qquad , \qquad \gamma_1 \;=\; 1 \qquad , \tag{27}$$

and, clearly, $\alpha_d / \gamma_d = p/r$. Finally, let F_t be the matrix

$$F_t \;=\; XY^{z_t}XY^{z_{t-1}}X \cdots XY^{z_1}X \qquad ; \tag{28}$$

by definition, one has $F_d = F(p/r)$.

Lemma 1

The matrix element $(F_t)_{mn}$ is given by

$$(F_t)_{mn} \;=\; B_t \sum_{s(m,k,|\alpha_t|)} \left[e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s+\frac{n}{\gamma_t}\right)^2} \;-\; e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s-\frac{n}{\gamma_t}\right)^2} \right] \qquad ; \tag{29}$$

$$B_t \;=\; \frac{(-i)^{t+1}}{\sqrt{2k|\alpha_t|}}\, \xi^{z_1+z_2+\cdots+z_t}\, \exp\left\{ -\frac{i\pi}{4}\left[sign(\alpha_0\alpha_1) \;+\cdots+ sign(\alpha_{t-1}\alpha_t) \right] \right\}$$
$$\exp\left\{ \frac{i\pi n^2}{2k}\left[\frac{1}{\alpha_0\alpha_1} \;+\cdots+\; \frac{1}{\alpha_{t-2}\alpha_{t-1}} \right] \right\} \qquad ; \tag{30}$$

where $s(m,k,|\alpha_t|)$ stands for the sum over a complete residue system modulo $(2k|\alpha_t|)$ with the additional constraint $s \equiv m \; (mod\, 2k)$.

Proof

The proof is based on induction. First of all we need to verify the validity of equations (29) and (30) when $t = 1$. In this case, from the definition (28) one gets

$$(F_1)_{mn} \;=\; -\frac{1}{2k}\,\frac{1}{2}\,\xi^{z_1} \sum_{s=1}^{2k} e^{-i\pi s^2 z_1/(2k)} \left[e^{-i\pi s(m+n)/k} \;-\; e^{-i\pi s(m-n)/k} \;+\; \text{c. c.} \right] \quad . \tag{31}$$

Since the sum (31) covers twice a complete residue system modulo k, i.e. $1 \leq s \leq 2k$, a multiplicative factor $1/2$ has been introduced in (31). The change of variables $s \to -s$ shows that the last two terms in (31) are equal to the first two terms. Therefore, equation (31) can be written as

$$(F_1)_{mn} = -\frac{1}{2k}\,\xi^{z_1} \sum_{s=1}^{2k} e^{-i\pi s^2 z_1/(2k)} \left[e^{-i\pi s(m+n)/k} - e^{-i\pi s(m-n)/k} \right] \quad . \tag{32}$$

At this point, one can use the reciprocity formula [19] reported in the appendix and one gets

$$(F_1)_{mn} = \frac{-1}{\sqrt{2k|z_1|}}\,\xi^{z_1} \exp\left\{ -\frac{i\pi}{4}\, sign(\alpha_0\alpha_1) \right\} \times$$
$$\times \sum_{v=0}^{|z_1|-1} \left[e^{\frac{i\pi}{2kz_1}(2kv+m+n)^2} - e^{\frac{i\pi}{2kz_1}(2kv+m-n)^2} \right] \quad . \tag{33}$$

By introducing the new variable $s = 2kv + m$, one finds that in equation (33) the variable s covers a complete residue system modulo $(2k|z_1|)$ with the constraint that $s \equiv m\ (mod\, 2k)$. Therefore, equation (33) can be written in the form

$$(F_1)_{mn} = B_1 \sum_{s(m,k,|z_1|)} \left[e^{\frac{i\pi}{2kz_1}(s+n)^2} - e^{\frac{i\pi}{2kz_1}(s-n)^2} \right] \quad . \tag{34}$$

This confirms the validity of equation (29) when $t = 1$. In order to complete the proof, suppose now that (29) is true for a given t; we shall show that (29) is true also in the case $t + 1$. Indeed, one has

$$(F_{t+1})_{mn} = \sum_{v=1}^{k} (\, X\, Y^{z_{t+1}}\,)_{mv}\ (\, F_t\,)_{vn} \quad . \tag{35}$$

From equation (29) one gets

$$(F_{t+1})_{mn} = -B_t \frac{i\xi^{z_{t+1}}}{\sqrt{2k}}\,\frac{1}{2} \sum_{v=1}^{2k} \sum_{s(v,k,|\alpha_t|)} e^{-\frac{i\pi}{2k}v^2 z_{t+1}}$$
$$\Bigg[e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s-\frac{n}{\gamma_t}\right)^2} e^{-i\pi mv/k} - e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s+\frac{n}{\gamma_t}\right)^2} e^{-i\pi mv/k}$$
$$- e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s-\frac{n}{\gamma_t}\right)^2} e^{i\pi mv/k} + e^{\frac{i\pi\gamma_t}{2k\alpha_t}\left(s+\frac{n}{\gamma_t}\right)^2} e^{i\pi mv/k} \Bigg] \quad . \tag{36}$$

Again, the last two terms can be omitted provided one introduces a multiplicative factor 2. Moreover, because of the constraint $v = s\,(mod\, 2k)$, one can set $v = s$, thus

$$(F_{t+1})_{mn} = -B_t \frac{i\,\xi^{z_{t+1}}}{\sqrt{2k}}\, e^{\frac{i\pi n^2}{2k\alpha_t\gamma_t}} \sum_{s=0}^{2k|\alpha_t|-1}$$
$$\left\{ e^{-\frac{i\pi}{2k\alpha_t}\left[\alpha_{t+1}s^2+2(\gamma_{t+1}m+n)s\right]} - e^{-\frac{i\pi}{2k\alpha_t}\left[\alpha_{t+1}s^2+2(\gamma_{t+1}m-n)s\right]} \right\} \quad . \tag{37}$$

By using the reciprocity formula, one obtains the final expression for $(F_{t+1})_{mn}$

$$(F_{t+1})_{mn} \;=\; -i\, B_t\, \xi^{z_{t+1}} \sqrt{\frac{|\alpha_t|}{|\alpha_{t+1}|}}\; e^{\left(\frac{i\pi n^2}{2k\alpha_t\alpha_{t+1}}\right)}\; e^{\frac{-i\pi}{4} sign(\alpha_t\alpha_{t+1})}$$
$$\sum_{v=1}^{|\alpha_{t+1}|} \left\{ e^{\frac{i\pi\alpha_t}{2k\alpha_{t+1}}\left(2kv+m+\frac{n}{\alpha_t}\right)^2} - e^{\frac{i\pi\alpha_t}{2k\alpha_{t+1}}\left(2kv+m-\frac{n}{\alpha_t}\right)^2} \right\} \quad . \tag{38}$$

In terms of the variable $s = 2kv + m$, equation (38) can be rewritten in the form (29) and this concludes the proof. ♠

From the definition (19) and **Lemma 1** it follows

Theorem 1

Let $SU(2)$ be the gauge group, the 3-manifolds invariant $I_k(L_{p/r})$ for $k \geq 2$ is given by

$$I_k(L_{p/r}) \;=\; \sum_{s\,(mod\;p)} \left\{ \exp\left[\frac{i\pi(r+1)^2}{2pkr}\right] \exp\left[\frac{i2\pi}{p}\left[rks^2 \,+\, (r+1)s\right]\right] \right.$$
$$\left. -\; \exp\left[\frac{i\pi(r-1)^2}{2pkr}\right] \exp\left[\frac{i2\pi}{p}\left[rks^2 \,+\, (r-1)s\right]\right] \right\} \frac{e^{i\theta_k \sigma(\mathcal{L})}\, B_d}{a_k} \quad . \tag{39}$$

Proof

According to equation (19), the expression for the matrix element $[F(p/r)]_{11}$ has been written by means of a sum over a complete residue system modulo p. ♠

As shown in equation (39), the expression for $I_k(L_{p/r})$ is rather involved; nevertheless, $|\,I_k(L_{p/r})\,|^2$ can be computed explicitly. Let us introduce the modulo-p Croneker delta symbol defined by

$$\delta_p\,(x) \;=\; \begin{cases} 0 & x \not\equiv 0 \;\;(\,mod\,p\,) \\ 1 & x \equiv 0 \;\;(\,mod\,p\,) \end{cases} \quad ; \tag{40}$$

where p and x are integers. One can easily verify that, for integer n,

$$\begin{cases} \delta_p(xn) \;=\; \delta_p(x) & \text{if} \quad (n,p) \,=\, 1 \quad ; \\ \delta_{pn}(xn) \;=\; \delta_p(x) & . \end{cases} \tag{41}$$

Finally, we shall denote by $\phi(n)$ the Euler function [20] which is equal to the number of residue classes modulo n which are coprime with n.

Theorem 2

The square of the absolute value of $I_k(L_{p/r})$ is given in the following list ;
for $p = 2$

$$\left|\, I_k(L_{2/1})\,\right|^2 \;=\; \left[\,1 + (-1)^k\,\right] \frac{\sin^2\left[\pi/(2k)\right]}{\sin^2\left[\pi/k\right]} \quad ; \tag{42}$$

for $p > 2$ one has :

when p *and* k *are coprime integers, i.e.* $(k,p) = 1$,

$$\Big| I_k(L_{p/r}) \Big|^2 = \frac{1}{2}\,[\,1-(-1)^p\,]\,\frac{\sin^2\Big[\pi\Big(k^{\phi(p)}-1\Big)/(kp)\Big]}{\sin^2(\pi/k)} + \frac{1}{2}\,[\,1+(-1)^p\,]\,\Big[\,1+(-1)^{p/2}\Big]\,\frac{\sin^2\Big[\pi\Big(k^{\phi(p/2)}-1\Big)/(kp)\Big]}{\sin^2(\pi/k)} \qquad ; \qquad (43)$$

when the greatest common divisor of p *and* k *is greater than unity, i.e.* $(k,p) = g > 1$ *and* p/g *is odd*

$$\Big| I_k(L_{p/r}) \Big|^2 = \frac{g}{4\,\sin^2(\pi/k)}\,[\,\delta_g(\,r-1\,) + \delta_g(\,r+1\,)\,] \qquad ; \qquad (44)$$

when $(k,p) = g > 1$ *and* p/g *is even*

$$\Big| I_k(L_{p/r}) \Big|^2 = \frac{g}{4\,\sin^2(\pi/k)}\,\Big\{\,\delta_g(r+1)\,\Big[\,1+(-1)^{kp/2g^2}\,(-1)^{(r+1)/g}\Big] + \delta_g(r-1)\,\Big[\,1+(-1)^{kp/2g^2}\,(-1)^{(r-1)/g}\Big]\Big\} \qquad . \qquad (45)$$

Proof
From Theorem 1 it follows that the square of the absolute value of the lens space invariant is

$$\Big| I_k(L_{p/r}) \Big|^2 = a(k)^{-2}\,(2kp)^{-1}\,\mathcal{S}(k,p,r) \qquad , \qquad (46)$$

with

$$\begin{aligned}\mathcal{S}(k,p,r) = \sum_{s,t\,(mod\,p)} \Big\{ &\exp\Big\{\frac{i2\pi}{p}\Big[kr\Big(s^2-t^2\Big) + (r+1)\,(s-t)\Big]\Big\} \\ &- \exp\Big(\frac{i2\pi}{kp}\Big)\,\exp\Big\{\frac{i2\pi}{p}\Big[kr\Big(s^2-t^2\Big) + r\,(s-t) + s + t\Big]\Big\} \\ &- \exp\Big(-\frac{i2\pi}{kp}\Big)\,\exp\Big\{\frac{i2\pi}{p}\Big[kr\Big(s^2-t^2\Big) + r\,(s-t) - s - t\Big]\Big\} \\ &+ \exp\Big\{\frac{i2\pi}{p}\Big[kr\Big(s^2-t^2\Big) + (r-1)\,(s-t)\Big]\Big\}\Big\} \qquad . \qquad (47)\end{aligned}$$

The indices s and t run over a complete residue system modulo p. When $p = 2$, each sum contains only two terms and the evaluation of (47) is straightforward; the corresponding result is shown in equation (42). Let us now consider the case in which $p > 2$. By means of the change of variables $s \to s + t$, the sum in t becomes a geometric sum and one obtains

$$\mathcal{S}(k,p,r) = p \sum_{s\,(mod\,p)} \Big\{\exp\Big\{\frac{i2\pi}{p}\Big[krs^2 + (r+1)\,s\Big]\Big\}\,\delta_p\,(2krs)$$

$$
\begin{aligned}
&- \exp\left(\frac{i2\pi}{kp}\right) \exp\left\{\frac{i2\pi}{p}\left[krs^2 + (r+1)\,s\right]\right\} \delta_p(2krs + 2) \\
&- \exp\left(\frac{-i2\pi}{kp}\right) \exp\left\{\frac{i2\pi}{p}\left[krs^2 + (r-1)\,s\right]\right\} \delta_p(2krs - 2) \\
&+ \exp\left\{\frac{i2\pi}{p}\left[krs^2 + (r-1)\,s\right]\right\} \delta_p\left(2krs\right)\Bigg\} \qquad . \qquad (48)
\end{aligned}
$$

By using properties (41), one can determine the values of s which give contribution to (48). Let us start with $(k,p) = 1$. Clearly, in this case one has

$$
\delta_p(2rks) \neq 0 \Rightarrow \begin{cases} s = p & p \text{ odd} \\ s = p,\, p/2 & p \text{ even} \end{cases} \qquad . \qquad (49)
$$

When $(k,p) = 1$ and p is odd, one gets

$$
\delta_p(2krs \mp 2) = \delta_p(krs \mp 1) \qquad . \qquad (50)
$$

The delta gives a non-vanishing contribution if and only if the following congruence is satisfied

$$
rks = \pm 1 \ \ (mod\, p) \quad . \qquad (51)
$$

The unique solution [20] to (51) is given by

$$
s = \pm (rk)^{\phi(p)-1} \qquad . \qquad (52)
$$

When $(k,p) = 1$ and p is even, one finds two solutions

$$
s_1 = \pm (rk)^{\phi(p/2)-1} \quad , \quad s_2 = \pm (rk)^{\phi(p/2)-1} + p/2 \qquad . \qquad (53)
$$

Let us now examine the case $(p,k) = g > 1$. We introduce the integer β defined by $p = g\beta$. For β odd, one has

$$
\delta_p(2krs) = \delta_\beta(s) \qquad . \qquad (54)
$$

Within the residues of a complete system modulo p, the values of s giving non-vanishing contribution are of the form $s = \alpha\beta$ with $1 \leq \alpha \leq g$. When β is even, one gets

$$
\delta_p(2krs) = \delta_\beta(2s) = \delta_{\beta/2}(s) \qquad . \qquad (55)
$$

The solutions of the associated congruence are

$$
s = \alpha \frac{\beta}{2} \qquad 1 \leq \alpha \leq 2g \qquad . \qquad (56)
$$

When $(k,p) = g > 1$ and p is odd, $\delta_p\left[2r(ks \pm 1)\right]$ does not contribute because $rks = \pm 1\ (mod\, p)$ has no solutions. On the other hand, if p is even we have

$$
\delta_p\left[\,(2rks \pm 2)\,\right] = \delta_{p/2}(rks \pm 1) \qquad . \qquad (57)
$$

The delta function (57) is non-vanishing when $(p/2, k) = 1$ and, in this case, the two solutions are $s_1 = \pm (rk)^{\phi(p/2)-1}$ and $s_2 = s_1 + p/2$. This exhausts the analysis of the modulo p Croneker deltas when $p > 2$.

At this stage, Theorem 2 simply follows from the substitution of the values of s for which the various Croneker deltas modulo p are non vanishing. In the case $(k,p) = 1$ and p odd, the algebraic manipulations are straightforward. When $(k,p) = 1$ and p even, the evaluation of (48) needs some care. In this case, one has to deal with factors of the form

$$\exp\left[\frac{i\pi}{b}\left(a^{\phi(b)} - 1\right)\right] \qquad ; \tag{58}$$

with $b > 2$ even and $(a,b) = 1$. In appendix B, it is shown that terms of the type (58) are trivial because actually

$$a^{\phi(b)} \equiv 1 \quad (\, mod\ 2b) \qquad . \tag{59}$$

Finally, the derivation of equations (44) and (45) is straightforward. ♠

Let us now consider the dependence of $|\, I(L_{p/r})\,|^2$ on r. As shown in equations (44) and (45), $|\, I(L_{p/r})\,|^2$ depends on r. However, this dependence is rather peculiar: when $I(L_{p/r}) \neq 0$, $|\, I(L_{p/r})\,|^2$ does not depend on r. Indeed, when expression (44) is different from zero, its values are given by

$$0 \;\neq\; (44) \;=\; \begin{cases} \sin^{-2}(\pi/k) & \text{for } g = 2\ ; \\ (g/4)\ \sin^{-2}(\pi/k) & \text{for } g > 2\ . \end{cases} \tag{60}$$

Similarly, when expression (45) is different from zero, its value is given by

$$0 \;\neq\; (45) \;=\; \frac{g}{2\, \sin^2(\pi/k)} \qquad . \tag{61}$$

To sum up, when $I(L_{p/r}) \neq 0$, $|\, I(L_{p/r})\,|^2$ only depends on p and, therefore, it is a function of the fundamental group $\pi_1(L_{p/r}) = Z_p$. Thus, Theorem 2 proves the validity of our conjecture for the lens spaces when the gauge group is $SU(2)$.

5 The SU(3) case

In this section we shall present numerical computations confirming the validity of our conjecture for lens spaces when the gauge group is $SU(3)$. As in the $SU(2)$ case, the $SU(3)$ Chern-Simons field theory can be solved explicitly in any closed, connected and orientable three-manifold [16]. The general surgery rules for $SU(3)$ and for any integer k have been derived in [16]. In particular, it turns out that

$$I_{k=1}(L_{p/r}) \;=\; \left[\, I_{k=2}(L_{p/r})\,\right]^* \;=\; I_{k=4}(L_{p/r}) \qquad . \tag{62}$$

Therefore, we only need to consider the case $k \geq 3$. For $k \geq 3$, the matrices which give a projective representation of the modular group have the following form

$$X_{(m,n)\,(a,b)} \;=\; \frac{i}{k\sqrt{3}} q^{-2} q^{-[(m+n)(a+b+3)+(m+3)b+(n+3)a]/3}$$

$$\Big[1 + q^{(n+1)(a+b+2)+(m+1)(b+1)} + q^{(m+1)(a+b+2)+(n+1)(a+1)}$$
$$- q^{(m+1)(b+1)} - q^{(n+1)(a+1)} - q^{(m+n+2)(a+b+2)} \Big] \quad ; \tag{63}$$

$$Y_{(a,b)\,(m,n)} = q^{[m^2+n^2+mn+3(m+n)]/3} \delta_{am}\, \delta_{bn} \quad ; \tag{64}$$

$$C_{(a,b)\,(m,n)} = \delta_{an}\delta_{bm} \quad ; \tag{65}$$

where each irreducible representation of $SU(3)$ has been denoted by a couple of nonnegative integers (m,n) (Dynkin labels).
By using equation (18), we have computed $I_k(L_{p/r})$ numerically for some examples of lens spaces. In particular, we have worked out the value of the invariant for the lens spaces $L_{p/r}$, with $p \leq 20$ and $3 \leq k \leq 50$. In all these cases, the results are in agreement with our conjecture.

Our calculations have been performed on a Pentium based PC running Linux. For instance, the results of the computations for the cases $L_{8/1}$, $L_{8/3}$, $L_{15/1}$, $L_{15/2}$, $L_{15/4}$ with $3 \leq k \leq 50$ are shown in Tables 1, 2, 3, 4, 5. The spaces $L_{8/1}$ and $L_{8/3}$ are not homotopically equivalent; as shown in Tables 1 and 2, the phase of the invariant distinguishes these two manifolds. The case in which $p = 15$ is more interesting because there are two different spaces belonging to the same homotopy class; $L_{15/1}$ and $L_{15/4}$ are homotopically equivalent and $L_{15/2}$ represents the other homotopy class. The phase of the invariant distinguishes the manifolds of the same homotopy class.

6 Conclusions

In this article, we have presented some arguments and numerical results supporting the conjecture that, for nonvanishing $I(M)$, the absolute value $|\,I(M)\,|$ only depends on the fundamental group $\pi_1(M)$. Since the Turaev-Viro invariant [21] coincides [3] with $|\,I(M)\,|^2$, our conjecture gives some hints on the topological interpretation of the Turaev-Viro invariant. For the gauge group $SU(2)$, $|\,I(M)\,|^2$ can be understood as the improved partition function of the Euclidean version of (2+1) gravity with positive cosmological constant [22, 23]. In this case, our conjecture suggests that, for instance, the semiclassical limit is uniquely determined by the fundamental group of the universe.

Finally, one may ask for which values of k the equality $I_k(M) = 0$ is satisfied and what the meaning of this fact is. The complete solution to this problem is not known. From the field theory point of view, gauge invariance of the factor $\exp(iS_{CS})$ (where S_{SC} is the Chern-Simons action) in the functional measure gives nontrivial constraints on the admissible values of k in a given manifold M. In certain cases [9] one finds that, in correspondence with the "forbidden" values of k, the invariant $I_k(M)$ vanishes. So, it is natural to expect that $I_k(M) = 0$ is related to a breaking of gauge invariance for large gauge transformations. From the mathematical point of view, $I_k(M) = 0$ signals the absence of the natural extension of $E(\mathcal{L})$ to an invariant $E_M(\mathcal{L})$ of links in the manifold M. More precisely, when $I_k(M) \neq 0$ for fixed integer k, one can define [13] an invariant $E_M(\mathcal{L})$ of oriented, framed and

coloured links $\{ \mathcal{L} \subset M \}$ with the following property: if the link $\mathcal{L}$ belongs to a three-ball embedded in M, then one has $E_{_M}(\mathcal{L}) = E(\mathcal{L})$. The values of the invariant $E_{_M}(\mathcal{L})$ correspond to the vacuum expectation values of the Wilson line operators associated with links in the manifold M. When $I_k(M) = 0$, the invariant $E_{_M}(\mathcal{L})$ cannot be constructed; consequently, for these particular values of k, the quantum Chern-Simons field theory is not well defined in M.

Acknowledgments. We wish to thank Turaev for useful discussions.

Appendix A

The generalized Gauss sums have a very useful property which can be expressed by means of the so-called reciprocity formula [19]

$$\sum_{n=0}^{|c|-1} e^{i\frac{\pi}{c}(an^2+bn)} = \sqrt{\left|\frac{c}{a}\right|}\, e^{i\frac{\pi}{4ac}(|ac|-b^2)} \sum_{n=0}^{|a|-1} e^{-i\frac{\pi}{a}(cn^2+bn)} \qquad , \tag{66}$$

where the integers a, b, c satisfy the relations

$$ac \neq 0 \qquad , \qquad ac + b \text{ is even} \qquad . \tag{67}$$

Appendix B

Lemma 2 *Let a, b two integers, with $(a,b) = 1$ and $b > 2$ even; one has*

$$a^{\phi(b)} \equiv 1 \quad (\mathit{mod}\ 2b) \qquad . \tag{68}$$

Proof

The proof consists of two parts: firstly, it is shown by induction that Lemma 2 holds when $b = 2^m$ with $m > 1$ integer. Secondly, equation (68) is proved when $b = 2^m c$ with $m \geq 1$ and c odd integer.

Since b is even, a is clearly odd and can be written in the form $a = (2f + 1)$. When b is of the type $b = 2^m$, the condition $b > 2$ implies that $m \geq 2$. Let us now consider the case $m = 2$; one has $\phi(b) = \phi(\, 2^2\,) = 2$, therefore

$$a^{\phi(b)} = (2f + 1)^2 = 1 + 4f(\, f + 1\,) \equiv 1 \quad (\, \mathit{mod}\ 2^3\,) \qquad . \tag{69}$$

Thus, **Lemma 2** is satisfied when $b = 2^2$. Suppose now that equation (68) holds when $b = 2^n$ for a certain n. We need to prove that (68) is true also for $b = 2^{(n+1)}$. Indeed, $\phi(2^{n+1}\,) = 2^n$ and one gets

$$(2f + 1)^{\phi(2^{n+1})} = \Big[(2f + 1)^{\phi(2^n\,)} \Big]^2 \qquad . \tag{70}$$

By using the induction hypothesis

$$(2f \;+\; 1)^{\phi(2^n\,)} \;=\; 1 \,+\, N\,2^{n+1} \qquad , \tag{71}$$

one finds

$$\Big[\,(2f \;+\; 1)^{\phi(2^n\,)}\,\Big]^2 \;=\; 1 \,+\, 2^{n+2}\,N\,\left(\,1 \,+\, 2^n\,N\,\right) \;\equiv\; 1 \quad (\,mod\; 2^{n+2}\,)\;. \tag{72}$$

Therefore, equation (68) is also satisfied when $b \,=\, 2^{(n+1)}$. To sum up, for $m > 1$ and a odd, one has

$$a^{\phi(2^m)} \;\equiv\; 1 \quad (\,mod\; 2^{m+1}\,) \qquad . \tag{73}$$

Let us now consider the general case in which $b = 2^m\,c$ with c odd integer. From Euler Theorem [20] it follows that

$$a^{\phi(b)} \;\equiv\; 1 \quad (\,mod\; b) \quad\;\; \Rightarrow \quad\;\; a^{\phi(b)} \;\equiv\; 1 \quad (\,mod\; c) \quad . \tag{74}$$

On the other hand, $\phi(2^m c\,) = \phi(2^m\,)\phi(c)$ and, for $m > 1$, equation (73) implies

$$a^{\phi(b)} \;=\; a^{\phi(c)\,\phi(2^m\,)} \;\equiv\; 1 \quad (mod\; 2^{m+1}) \qquad . \tag{75}$$

Since $(2^{m+1}, c) = 1$, from equations (74) and (75) one gets

$$a^{\phi(b)} \;\equiv\; 1 \quad (mod\; 2^{m+1}\,c) \;\equiv\; 1 \quad (mod\; 2b) \qquad . \tag{76}$$

Finally, we need to consider the case $b = 2\,c$. Since $\phi(c)$ is even, one gets

$$a^{\phi(2c)} \;=\; [\,1 \,+\, 4f(f+1)\,]^{\phi(c)/2} \;\equiv\; 1 \quad (\,mod\; 2^2\,) \qquad . \tag{77}$$

Equations (73) and (77) imply

$$a^{\phi(2c)} \;\equiv\; 1 \quad (\,mod\; 2^2 c\,) \qquad . \tag{78}$$

This concludes the proof. ♠

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•	$L_{8/1}$	•
k	I_k	$\|I_k\|$
3	1.000000000 - i 0.000000175	1.000000000
4	-1.000000000 + i 0.000000012	1.000000000
5	-0.499999839 + i 1.538841821	1.618033989
6	-2.000000000 + i 0.000000175	2.000000000
7	-1.000000000 - i 0.000000095	1.000000000
8	-6.828427084 + i 6.828427165	9.656854249
9	-0.499999950 + i 0.866025433	1.000000000
10	-4.236067816 + i 3.077683759	5.236067977
11	-2.073846587 - i 14.423920506	14.572244935
12	-7.464102180 - i 12.928202904	14.928203230
13	-12.373524802 - i 17.926145664	21.781891892
14	18.195669358 + i 0.000000868	18.195669358
15	5.657005398 + i 4.110054701	6.992443043
16	63.431390926 + i 26.274142180	68.657642707
17	6.721172941 + i 2.603796085	7.207906752
18	15.581719525 + i 26.988328071	31.163437478
19	-69.185356387 - i 11.544994773	70.142001987
20	-66.118464248 + i 0.000001734	66.118464248
21	-90.016155013 + i 0.000000715	90.016155013
22	43.367008373 - i 50.048195295	66.223253224
23	22.973052202 - i 3.157573698	23.189036183
24	219.054514271 - i 58.695485329	226.781922164
25	23.000602198 - i 5.905551129	23.746646829
26	84.434763344 + i 44.314782640	95.357376335
27	-185.409752744 + i 67.483626877	197.308936212
28	-161.491033989 + i 77.769978391	179.241523084
29	-214.342425435 + i 99.165358161	236.170369862
30	50.544948881 - i 155.561366312	163.566899299
31	47.834849646 - i 26.550506056	54.709251617
32	443.615385766 - i 296.414325177	533.531688523
33	46.894822945 - i 30.137471278	55.743982582
34	211.662329441 + i 39.566544008	215.328709440
35	-344.610722364 + i 250.374335362	425.962272715
36	-290.376668930 + i 243.654963011	379.059824907
37	-381.345093602 + i 307.914657384	490.138262785
38	27.064369107 - i 326.618342223	327.737732877
39	79.852315000 - i 70.742976321	106.681586554
40	734.287201006 - i 734.287220264	1038.438931957
41	78.053570702 - i 75.119028091	108.329258655
42	408.590935390 - i 0.000001624	408.590935390
43	-545.541472393 + i 565.843251992	785.998781122
44	-452.077098514 + i 521.724781996	690.340822456
45	-590.050989168 + i 655.318028318	881.817377951
46	-39.324647251 - i 574.905929557	576.249299975
47	118.947578576 - i 140.691854636	184.235513433
48	1090.316030520 - i 1420.927547540	1791.039960963
49	116.363242798 - i 145.914887153	186.632147733
50	687.196328608 - i 86.813088226	692.658145364

•	$L_{8/3}$	•
k	I_k	$\|I_k\|$
3	1.000000000 + i 0.000000000	1.000000000
4	-1.000000000 + i 0.000000000	1.000000000
5	1.309016994 - i 0.951056516	1.618033989
6	-2.000000000 + i 0.000000000	2.000000000
7	-0.623489802 + i 0.781831482	1.000000000
8	0.000000000 + i 0.000000000	0.000000000
9	-0.500000000 + i 0.866025404	1.000000000
10	1.618033989 + i 4.979796570	5.236067977
11	6.053529319 - i 13.255380237	14.572244935
12	-7.464101615 + i 12.928203230	14.928203230
13	7.723965314 - i 20.366422715	21.781891892
14	-16.393731622 + i 7.894805057	18.195669358
15	-2.160783733 + i 6.650208521	6.992443043
16	0.000000000 + i 0.000000000	0.000000000
17	-1.972537314 + i 6.932749548	7.207906752
18	15.581718739 + i 26.988328525	31.163437478
19	17.218843527 - i 67.995675380	70.142001987
20	-20.431729095 + i 62.882396270	66.118464248
21	20.030478885 - i 87.759262069	90.016155013
22	-55.710545730 + i 35.802993757	66.223253224
23	-4.717948848 + i 22.704016336	23.189036183
24	0.000000000 + i 0.000000000	0.000000000
25	-4.449677900 + i 23.326028428	23.746646829
26	54.169163837 + i 78.477582217	95.357376335
27	34.262337211 - i 194.311370120	197.308936212
28	-39.884991120 + i 174.747563877	179.241523084
29	38.208113963 - i 233.059184818	236.170369862
30	-132.328401250 + i 96.142211171	163.566899299
31	-8.284500381 + i 54.078362271	54.709251617
32	0.000000000 + i 0.000000000	0.000000000
33	-7.933195866 + i 55.176589215	55.743982582
34	129.764538515 + i 171.836019661	215.328709440
35	57.178306982 - i 422.107212669	425.962272715
36	-65.823047822 + i 373.301054424	379.059824907
37	62.256293514 - i 486.168356194	490.138262785
38	-258.631121471 + i 201.300681961	327.737732877
39	-12.859044288 + i 105.903757675	106.681586554
40	0.000000000 + i 0.000000000	0.000000000
41	-12.423570453 + i 107.614511930	108.329258655
42	254.752281347 + i 319.449256739	408.590935390
43	85.965636475 - i 781.283554973	785.998781122
44	-98.245742501 + i 683.314148273	690.340822456
45	92.175015400 - i 876.986690089	881.817377951
46	-447.003088251 + i 363.663986140	576.249299975
47	-18.441181139 + i 183.310248618	184.235513433
48	0.000000000 + i 0.000000000	0.000000000
49	-17.920983557 + i 185.769741658	186.632147733
50	441.516918550 + i 533.702273720	692.658145364

•	$L_{15/1}$	•
k	I_k	$\lvert I_k \rvert$
3	1.000000000 - i 0.000000175	1.000000000
4	0.000000021 + i 1.732050808	1.732050808
5	-2.665351925 + i 1.936491953	3.294556414
6	3.000000303 + i 3.464101353	4.582575695
7	-0.000000165 + i 1.732050808	1.732050808
8	-1.732050808 + i 0.000000010	1.732050808
9	-0.907604426 - i 11.866568847	11.901226911
10	-5.959909504 - i 18.342712091	19.286669182
11	-17.245203220 + i 14.943053456	22.818673947
12	-11.196152781 - i 8.196151933	13.875544804
13	0.000000084 - i 15.347547346	15.347547346
14	0.000000083 - i 1.732050808	1.732050808
15	-101.423006915 - i 32.954328677	106.642459228
16	-1.224744868 + i 1.224744875	1.732050808
17	-20.645987906 + i 15.591127116	25.871607244
18	-40.127945922 - i 30.808776018	50.590836361
19	65.239497521 - i 83.819706393	106.216454548
20	-115.811330427 - i 84.141852139	143.150674244
21	-53.061367118 - i 135.569663547	145.583798394
22	36.010672960 + i 16.445523443	39.588177633
23	-40.070955963 - i 2.740927872	40.164588848
24	164.290883844 - i 129.064834808	208.923972053
25	274.923532195 - i 34.730921473	277.108616721
26	0.000000762 - i 277.056941014	277.056941014
27	152.288761117 + i 30.720691710	155.356453557
28	-0.000003253 + i 136.433611353	136.433611353
29	-4.270422302 + i 12.674160517	13.374260781
30	485.145409358 + i 667.745345688	825.378649414
31	13.416605095 + i 0.680413518	13.433847358
32	164.960256203 + i 68.328775084	178.551694562
33	-82.729230504 + i 287.059281883	298.742626511
34	-538.981501170 - i 268.380890132	602.104111256
35	-415.659227492 + i 629.696787632	754.513510650
36	-681.771583909 + i 223.335013764	717.419696551
37	101.630605413 - i 150.367371847	181.491395038
38	-59.482939675 + i 173.267881065	183.193828283
39	-279.481438371 - i 847.784737546	892.663898458
40	347.379313920 - i 1069.123643311	1124.143119192
41	-941.842709163 - i 512.157840150	1072.088308877
42	399.545316356 - i 429.453517605	586.572061733
43	390.071778264 + i 279.072428091	479.622155784
44	24.267771316 + i 37.761389387	44.887049949
45	2753.539308039 + i 289.408610795	2768.706568944
46	32.920210846 - i 30.745331135	45.044596443
47	536.002877882 - i 206.437637985	574.382784800
48	394.055675294 + i 817.737030229	907.729985094
49	-1754.712300120 + i 400.501606784	1799.837990828
50	137.556258644 + i 2186.393963508	2190.716843400

•	$L_{15/2}$	•
k	I_k	$\|I_k\|$
3	1.000000000 + i 0.000000000	1.000000000
4	0.000000000 - i 1.732050808	1.732050808
5	0.000000000 + i 0.000000000	0.000000000
6	3.000000000 - i 3.464101615	4.582575695
7	0.000000000 - i 1.732050808	1.732050808
8	1.732050808 + i 0.000000000	1.732050808
9	10.730551990 - i 5.147276559	11.901226911
10	0.000000000 + i 0.000000000	0.000000000
11	-12.336706536 + i 19.196290072	22.818673947
12	-11.196152423 + i 8.196152423	13.875544804
13	14.350206054 - i 5.442315293	15.347547346
14	0.000000000 + i 1.732050808	1.732050808
15	0.000000000 + i 0.000000000	0.000000000
16	1.224744871 - i 1.224744871	1.732050808
17	9.345902508 + i 24.124555285	25.871607244
18	-6.617211192 - i 50.156208387	50.590836361
19	50.553444640 - i 93.414583721	106.216454548
20	0.000000000 + i 0.000000000	0.000000000
21	-53.061366041 + i 135.569663969	145.583798394
22	-11.153278505 + i 37.984578277	39.588177633
23	-29.353726021 - i 27.414466365	40.164588848
24	-164.290886665 - i 129.064831217	208.923972053
25	0.000000000 + i 0.000000000	0.000000000
26	228.013392388 + i 157.386281027	277.056941014
27	56.698638283 + i 144.640561664	155.356453557
28	0.000000000 - i 136.433611353	136.433611353
29	11.816320486 + i 6.264616638	13.374260781
30	0.000000000 + i 0.000000000	0.000000000
31	-1.359079795 - i 13.364922631	13.433847358
32	164.960256101 + i 68.328775330	178.551694562
33	-224.792217910 - i 196.762841162	298.742626511
34	-110.636340119 - i 591.852144574	602.104111256
35	0.000000000 + i 0.000000000	0.000000000
36	147.472006710 + i 702.099015977	717.419696551
37	45.731849880 + i 175.635202563	181.491395038
38	-177.588145856 - i 44.971426177	183.193828283
39	-875.291194999 - i 175.254556482	892.663898458
40	0.000000000 + i 0.000000000	0.000000000
41	1052.478012596 + i 204.116082250	1072.088308877
42	399.545318063 + i 429.453516017	586.572061733
43	-171.335849479 - i 447.974819607	479.622155784
44	44.430164502 + i 6.388093254	44.887049949
45	0.000000000 + i 0.000000000	0.000000000
46	-17.945816315 - i 41.315412930	45.044596443
47	571.498128245 + i 57.493242102	574.382784800
48	-817.737034535 - i 394.055666358	907.729985094
49	-780.920437266 - i 1621.597997004	1799.837990828
50	0.000000000 + i 0.000000000	0.000000000

•	$L_{15/4}$	•
k	I_k	$\|I_k\|$
3	1.000000000 + i 0.000000000	1.000000000
4	0.000000000 + i 1.732050808	1.732050808
5	-1.018073921 - i 3.133309346	3.294556414
6	3.000000000 + i 3.464101615	4.582575695
7	-0.751508681 - i 1.560523855	1.732050808
8	-1.732050808 + i 0.000000000	1.732050808
9	10.730551990 + i 5.147276559	11.901226911
10	-15.603243133 - i 11.336419711	19.286669182
11	17.245203123 + i 14.943053568	22.818673947
12	-11.196152423 - i 8.196152423	13.875544804
13	3.672908488 + i 14.901575513	15.347547346
14	-1.688624678 + i 0.385417563	1.732050808
15	0.000000000 + i 0.000000000	0.000000000
16	-1.224744871 + i 1.224744871	1.732050808
17	-17.429589094 - i 19.119348456	25.871607244
18	-6.617211192 + i 50.156208387	50.590836361
19	0.000000000 - i 106.216454548	106.216454548
20	-44.235991098 + i 136.144381552	143.150674244
21	72.909410760 - i 126.011349400	145.583798394
22	-36.010673013 + i 16.445523326	39.588177633
23	-10.836276386 + i 38.675176941	40.164588848
24	164.290886665 - i 129.064831217	208.923972053
25	-257.649075864 + i 102.010485575	277.108616721
26	275.036883975 - i 33.395523912	277.056941014
27	-153.611719961 + i 23.217819719	155.356453557
28	106.668092622 + i 85.064965309	136.433611353
29	-9.197471902 + i 9.709653035	13.374260781
30	0.000000000 + i 0.000000000	0.000000000
31	-4.021598499 + i 12.817761129	13.433847358
32	-164.960256101 - i 68.328775330	178.551694562
33	85.599713692 + i 286.216431937	298.742626511
34	-217.505092368 - i 561.445362956	602.104111256
35	33.851120653 + i 753.753765751	754.513510650
36	147.472006710 - i 702.099015977	717.419696551
37	-136.240563896 + i 119.906777214	181.491395038
38	0.000000000 + i 183.193828283	183.193828283
39	538.949599873 - i 711.605343156	892.663898458
40	-909.450887536 + i 660.754746927	1124.143119192
41	1008.985242455 - i 362.383943545	1072.088308877
42	-546.310790198 + i 213.568031595	586.572061733
43	426.548198677 + i 219.303548819	479.622155784
44	-24.267771377 + i 37.761389348	44.887049949
45	0.000000000 + i 0.000000000	0.000000000
46	-6.133571758 + i 44.625048641	45.044596443
47	-558.735830232 - i 133.153503483	574.382784800
48	394.055666358 + i 817.737034536	907.729985094
49	-883.212092863 - i 1568.232505801	1799.837990828
50	410.499402001 + i 2151.913225229	2190.716843400